

4.2 Application questions: other norms on finite-dimensional vectors

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1. You are at the origin in downtown Manhattan and you need to deliver two packages to two locations: $\mathbf{u}_1 = \begin{pmatrix} 134 \\ -31 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} -96 \\ 92 \end{pmatrix}$, and you can go to either first, but then you must proceed to the other. In this case, which would you visit first? Recall that you can only follow the streets, so you must use the 1-norm.
2. In Question 1, if you were able to use a drone to fly directly to either location, would you still visit the same location first?
3. The head of your 3-dimensional printer can be assumed to be currently at the origin. The strategy for printing is to minimize the time moving the head while not extruding a plastic. You are currently finished at the present position, and you can now travel to any one of the following five points relative to your current location to continue printing. Which do you choose? $\mathbf{u}_1 = \begin{pmatrix} 0.9 \\ 0 \\ 2.4 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0.9 \\ 2.4 \\ 2.5 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} -1.7 \\ -2.3 \\ -0.8 \end{pmatrix}$, $\mathbf{u}_4 = \begin{pmatrix} -2.3 \\ 1.7 \\ -4.6 \end{pmatrix}$, or $\mathbf{u}_5 = \begin{pmatrix} 3.2 \\ 0.3 \\ -0.3 \end{pmatrix}$? Recall that movement in each direction is controlled by an independent motor, so you must use the infinity-norm.

Comment and beyond the scope of this course: In Question 3, you may realize that always moving to the closest point may not result in the shortest overall possible path. You may have heard of the “travelling sales representative problem”, where one asks in what order can the sales representative visit n cities in order to minimize the overall distance travelled. If you were to find this optimal order in which to visit these five points, you would find you should visit them in the order $\mathbf{u}_2, \mathbf{u}_5, \mathbf{u}_1, \mathbf{u}_3$ and finally $\mathbf{u}_4$. This would result in an overall distance traveled of 15.2. If at each point, you chose to always go to the point closest to the current point, you would visit them in the order $\mathbf{u}_3$ (the point you should have found in the previous question), then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_5$ and finally $\mathbf{u}_4$, with the distance being 16.2, or approximately 6.6% further. This, however, introduces a fallacy (something that is incorrect): simply trying to get between the points in the fastest order only finds the order in which the distance travelled between those points is minimized. Because this is a 3-d printer, when it gets to a point, it starts to extrude plastic, and it may extrude plastic for some time while the head is moving, so it may end up far away from where it started. Thus, using the ∞ -norm only tells you how far apart the points are, but not necessarily how far the head will be moving while not extruding plastic. To optimize performance, you would have to use a completely different algorithm.

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